Dynamic Modeling of Structures from Measured Complex Modes

Samir R. Ibrahim*
Old Dominion University, Norfolk, Virginia

A technique is presented to use a set of identified complex modes together with an analytical mathematical model of a structure under test to compute improved mass, stiffness, and damping matrices. A set of identified normal modes, computed from the measured complex modes, is used in the mass orthogonality equation to compute an improved mass matrix. This eliminates possible errors that may result from using approximated complex modes as normal modes. The improved mass matrix, the measured complex modes, and the higher analytical modes are then used to compute the improved stiffness and damping matrices. The number of degrees-of-freedom of the improved model is limited to equal the number of elements in the measured modal vectors. A simulated experiment shows considerable improvements, in the system's analytical dynamic model, over the frequency range of the given measured modal information.

Nomenclature

$[C_A]$ = $(n \times n)$ analytical damping matrix
$[C_E]$ = $(n \times n)$ exact damping matrix
$[C]$ = $(n \times n)$ improved damping matrix
j = $\sqrt{-1}$
$[K_A]$ = $(n \times n)$ analytical stiffness matrix
$[K_E]$ = $(n \times n)$ exact stiffness matrix
$[K]$ = $(n \times n)$ improved stiffness matrix
m = number of measured modes
$[m_A]$ = $(n \times n)$ analytical modal mass matrix (diagonal)
$[M_A]$ = $(n \times n)$ analytical mass matrix
$[M_E]$ = $(n \times n)$ exact mass matrix
$[M]$ = $(n \times n)$ improved mass matrix
n = number of degrees-of-freedom of math models
$\{\phi_i\}$ = ith normal modal vector (n elements)
$\{\psi_i\}$ = ith complex modal vector (n elements)
ζ_i = ith damping factor
λ_i = ith characteristic root
ω_d = damped natural frequency
ω_n = natural frequency

Introduction

DUE to the increasing complexity of modern aerospace, and some nonaerospace structures, and due to the nature, sensitivity, and sophistication of the missions of such structures, an accurate mathematical model has become a necessity for successful performance. Such models are needed for responses and loads prediction, stability analysis, and control system design.

Past, and sometimes current, common practice, in spite of the advanced state-of-the-art in both finite element and structural dynamic identification, in arriving at a dependable mathematical model was done primarily by trial and error approach. An analyst, using some modal test data, adjusts his or her model, using personal judgment and experience, to make it fit the available modal test data. During the last three decades there have been continuous efforts by researchers and practitioners in the area of dynamic modeling of structures using identified modal parameters. The survey paper,[1] covering work done in the 1960's, pointed out a need to improve the state-of-the-art of dynamic modeling.

Presented as Paper 82-0770 at the AIAA/ASME/ASCE/AHS 23rd Structures, Structural Dynamics and Materials Conference, New Orleans, La., May 10-12, 1982; submitted May 12, 1982; revision received Sept. 17, 1982.

*Associate Professor, Department of Mechanical Engineering and Mechanics. Member AIAA.

Subsequent work in dynamic modeling from test data can be divided into two categories. The first category uses only experimental data to derive the mass, stiffness, and damping matrices.[2-4] The other category deals with using identified modal data to improve an existing, sometimes larger, analytical model.[5-8]

One of the important and basic relations often used in dynamic modeling is that the measured modes satisfy the theoretical requirement of weighted orthogonality with respect to the mass and stiffness matrices. Such a requirement can only be satisfied assuming no or proportional damping and a symmetrical stiffness matrix,[9] in which a case damped and normal modes are the same. For simpler structures the measured modes (complex modes) are very close to the normal modes. For more complex structures, the complex modes can be very much different from the normal modes. Attempts to use these complex modes, as normal modes, for satisfying the orthogonality requirement may lead to adverse effects on the process of dynamic modeling.

Complexity of modes, indicated by a scatter in the phase angles associated with the modal vector, is becoming more noticeable to today's dynamicist due to the complexity and damping characteristics of modern structures. Naturally, such a scatter in the phase angles could be due to measurement errors, erroneous identification, nonlinearities, as well as just the mere fact of having a case of complex modes as a result of the presence of nonproportional damping.

For structures with nonproportional damping, it is extremely difficult to measure normal modes even by using techniques such as multiple-sine-dwell, since this very technique is based on the assumption of proportional damping.[10] Using measured modes directly in the equation of orthogonality requirement can result in large errors in the off-diagonal terms.[11,12] Such errors can be due to the fact that the structure has complex modes (nonproportional damping) among other reasons.

The approach proposed herein is designed to circumvent using complex modes as normal modes, when correcting the analytical mass matrix. Instead, the procedure allows for the computation of normal modes from the given set of measured complex modes.

Theory and Procedure

In this procedure, it is assumed that the structure under consideration has an analytical mathematical model that needs improvements. Such a model can, as in most cases, be a finite element model. Furthermore, it is assumed that the structure has been tested in a modal survey test for the identification of its modal parameters. The following information is required for the procedure of improving the analytical model.

1) Modal Test Data. It is assumed that a complete modal survey test has been conducted and that the following test data are available: a) the complex modal vectors $\{\psi_i\}$, $i=1,...m$, measured at n measurement stations where $n>m$; b) the damped natural frequencies $(\omega_d)_i$, $i=1,...m$; and c) the damping factors ζ_i, $i=1,...m$.

2) Analytical Model Data. From the analytical model, the following information is required: a) an $n\times n$ mass matrix $[M_A]$ or the n elements of the modal mass matrix $[m_A]$; b) the normal modes $\{\phi_i\}$, $i=1,...n$, at the n measurement stations of the modal survey test; and c) the natural frequencies $(\omega_n)_i$, $i=1,...n$. The preceding information will be used to compute improved mass and stiffness matrices $[M]$ and $[K]$ and a damping matrix $[C]$.

Computation of $[M^{-1}K]$ and $[M^{-1}C]$

Assuming that the structure under consideration is linear, the measured modal parameters satisfy the following equation:

$$[M^{-1}K\ M^{-1}C]\begin{Bmatrix}\psi_i\\ \lambda_i\psi_i\end{Bmatrix}=\{-\lambda_i^2\psi_i\}\ldots\quad(i=1,2,...m)\tag{1}$$

where λ_i is the ith characteristic root of the system which is related to the ith damping factor and the ith damped natural frequency through the equation

$$\lambda_i=(\omega_d)_i\left\{\frac{\zeta_i}{\sqrt{1-\zeta_i^2}}+j\right\}\ldots\tag{2}$$

Equation (1) represents $n\times m$ complex equations or $2n\times m$ real equations. These equations are not sufficient to solve for system's $[M^{-1}KM^{-1}C]$.

Since no information is available to correct the analytical model beyond the frequency range over which the modal test was conducted, the analytical higher modes will be assumed to also satisfy Eq. (1). This will give the following set of equations:

$$[M^{-1}K\ M^{-1}C]\begin{Bmatrix}\phi_i\\ \lambda_i\phi_i\end{Bmatrix}=\{-\lambda_i^2\phi_i\}\ldots\quad(i=m+1,...n)\tag{3}$$

where λ_i in this case is:

$$\lambda_i=(\omega_n)_i\{-\zeta+j\sqrt{1-\zeta^2}\}\ldots\tag{4}$$

It is to be noted here that most analytical models do not have damping information. It is reasonable to assume that these higher analytical modes have a damping factor equal to the average damping factor of the m measured modes

$$\zeta=\frac{1}{m}\sum_{i=1}^{m}\zeta_i\tag{5}$$

Equation (3) represents $2n\times(n-m)$ equations. Combining Eqs. (1) and (3), the $2n^2$ linear equations can be solved for $[M^{-1}KM^{-1}C]$.

Computation of Experimental Normal Modes

The purpose of this section is to compute the set of normal modes, corresponding to the set of measured complex modes, for use in correcting the mass matrix. This step is essential in case the measured modes indicate the presence of non-proportional damping in the structure under test. This can be indicated clearly by large scatter in the phase angles associated with the measured modal vectors, a phenomenon found in several of today's modern complex structures. Such a complexity of measured modes is especially noticed in the higher modes that are needed badly to increase the frequency range of the dynamic model correction. It is the author's opinion that any effort to use complex modes, approximated as normal modes, to correct an analytical mass matrix may worsen rather than improve the analytical mass matrix.

Two approaches[13] were presented to compute normal modes from measured complex modes, one of which is similar to the method presented in the preceding section to compute $[M^{-1}KM^{-1}C]$. The $[M^{-1}K]$ matrix can yield a set of normal modes, corresponding to the set of measured complex modes, through the relation

$$[M^{-1}K]\{\phi\}=\omega^2\{\phi\}\ldots\tag{6}$$

This eigenvalue equation will give n eigenvalues and n eigenvectors. The first m of these eigenvectors are the m computed normal modes corresponding to m measured complex modes. The remainder eigenvectors will be the higher analytical modes used in Eq. (3).

Correction of the Mass Matrix

Several approaches can be used to correct the mass or stiffness matrices. The approach based on minimum changes,[8] gives the corrected mass matrix as

$$[m_A]=[\phi]^T[M_A][\phi]$$

$$[M]=[M_A]+[M_A][\phi][m_A]^{-1}[I-m_A][m_A]^{-1}[\phi]^T[M_A]$$

where $[\phi]$ is $(n\times m)$ "measured" normal modes, and $[m_A]$ is $m\times m$.

The approach used herein is simply based on computing a mass matrix that satisfies the orthogonality condition for the measured normal modes and the higher analytical normal modes, i.e.,

$$[\phi]^T[M][\phi]=[m_A]\tag{7}$$

where the columns of $[\phi]$ in this case are the eigenvectors computed from Eq. (6).

Computation of Corrected Stiffness and Damping Matrices

After computing $[M^{-1}KM^{-1}C]$ from Eqs. (1) and (3) and $[M]$ from Eq. (7), the stiffness and damping matrices can be given by

$$[K]=[M][M^{-1}K]\tag{8}$$

$$[C]=[M][M^{-1}C]\tag{9}$$

and this completes the computation of corrected or improved mass, stiffness, and damping matrices.

Criteria for Evaluating Dynamic Model Improvements

The question of judging the success of any dynamic model improvements technique is quite a difficult one. Should the changes to the analytical model be minimum? Should the improved model represent a physical system rather than just a set of numbers? What about ending with negative masses or negative stiffness in the improved mass matrix? The answer to the question of success should be very much dependent on the intended use of the improved model.

In the work reported here, the goal of improving the analytical mathematical model is to make the improved model respond to any input as close as possible to the response of the exact model (real structure) over the correction frequency range. This makes the improved model suitable for responses and loads prediction and control system design but not for structural modifications. If the improved model is to be used for structural modifications, the number of degrees-of-

Table 1 Exact, analytical, and improved frequencies and damping factors

	Frequency, Hz			Damping factor ζ, %			
Mode	Exact	Analytical	Improved	Exact	Analytical	Improved	Notes
1	9.999	10.082	9.998	2.00	1.00	2.00	Frequency
1	11.998	12.142	11.998	2.00	1.00	2.00	range of
3	14.997	15.204	14.997	2.00	1.00	2.00	modal test
4	19.996	19.254	19.996	2.00	1.00	2.00	data
5	23.995	23.911	23.911	2.00	1.00	1.00	
6	29.994	31.683	31.683	2.00	1.00	1.00	
7	35.993	33.740	33.740	2.00	1.00	1.00	
8	42.991	41.020	41.020	2.00	1.00	1.00	
9	45.991	44.293	44.293	2.00	1.00	1.00	
10	49.990	46.104	46.104	2.00	1.00	1.00	

Table 2 Exact, analytical, and improved mode shapes

	Exact		Analytical		Improved	
Mode	Amplitude	Phase, deg	Amplitude	Phase, deg	Amplitude	Phase, deg
	100.00	0.0	100.00	0.0	100.00	0.0
	157.00	−0.4	148.49	0.0	157.00	−6.4
	155.56	−19.7	141.85	0.0	155.56	−19.7
	116.55	−45.5	93.71	0.0	116.55	−45.5
	81.11	−93.2	29.97	0.0	81.11	−93.2
2	81.11	−156.7	29.06	180.0	81.11	−156.7
	116.55	155.6	95.00	180.0	116.55	155.6
	155.56	129.8	134.79	180.0	155.56	129.8
	157.00	116.5	141.71	180.0	157.00	116.5
	100.00	110.1	101.35	180.0	100.00	110.1
	100.00	0.0	100.00	0.0	100.00	0.0
	74.24	−18.0	70.32	0.0	74.24	−18.0
	51.66	−150.3	43.19	180.0	51.66	−150.3
	99.08	166.7	91.65	180.0	99.08	166.7
	63.70	116.5	44.89	180.0	63.70	116.5
4	63.70	6.4	43.48	0.0	63.70	6.4
	99.08	−43.8	89.33	0.0	99.08	−43.8
	51.66	−86.8	42.40	0.0	51.66	−86.8
	74.24	−140.9	72.75	180.0	74.24	140.9
	100.00	122.9	106.09	180.0	100.00	122.9

freedom of the analytical model should be larger than the number of elements in the measured modal vectors. This will require the computation of the unmeasured modal vectors' elements. That is a point to be considered for future investigations.

Illustrative Simulated Experiment

The purpose of selecting a simulated experiment, rather than a real experiment, is to test the effectiveness of the proposed technique under controlled conditions. In this simulated study an exact mathematical model is available as a reference for comparison. This exact mathematical model is corrupted with random errors to produce an analytical model which is to be corrected to produce the improved mathematical model. A comparison is later conducted between the improved, analytical, and exact mathematical models.

Exact Model

The exact model possesses ten degrees-of-freedom. It is derived through assuming ten complex modes of the form

$$\psi_{ik} = \sin\frac{ik\pi}{11} + j0.5\sin\frac{i(k+1)\pi}{11} \quad (i=1,2,\ldots 10 \text{ and } k=1,2,\ldots 10) \qquad (10)$$

The ten modes were assumed to have undamped natural frequencies of 10, 12, 15, 20, 24, 30, 36, 43, 46, and 50 and a damping factor of 2.0% for all ten modes.

Using the preceding modal information, $[M_E^{-1}K_E]$ and $[M_E^{-1}C_E]$ were computed. From $[M_E^{-1}K_E]$ the normal modes were computed and then used in the equation

$$[\phi]^T[M_E][\phi] = [M_A] \qquad (11)$$

with all elements of $[m_A]$ assumed equal 1.0×10^{-5}; $[M_E]$ and, subsequently, $[K_E]$ and $[C_E]$ were calculated.

Analytical Model

Random errors ranging between ±1.0% for first mode to ±10.0% for the tenth mode were introduced in the ten undamped natural frequencies. Random errors of ±5.0% were introduced to exact normal modes. These corrupted modal parameters were then used to calculate $[M_A^{-1}K_A\ M_A^{-1}C_A]$ for the analytical mathematical model with proportional damping equivalent to 1.0%. The modal mass matrix $[m_A]$ from the exact model was used with ±5.0% random errors to calculate $[M_A]$, $[K_A]$, and $[C_A]$.

Improved Model

Exact modal parameters (complex mode shapes, damped natural frequencies, and damping factors) of the first four modes here are considered as the measured modal parameters. These four modes together with the six higher analytical modes were used to correct the analytical model as previously described.

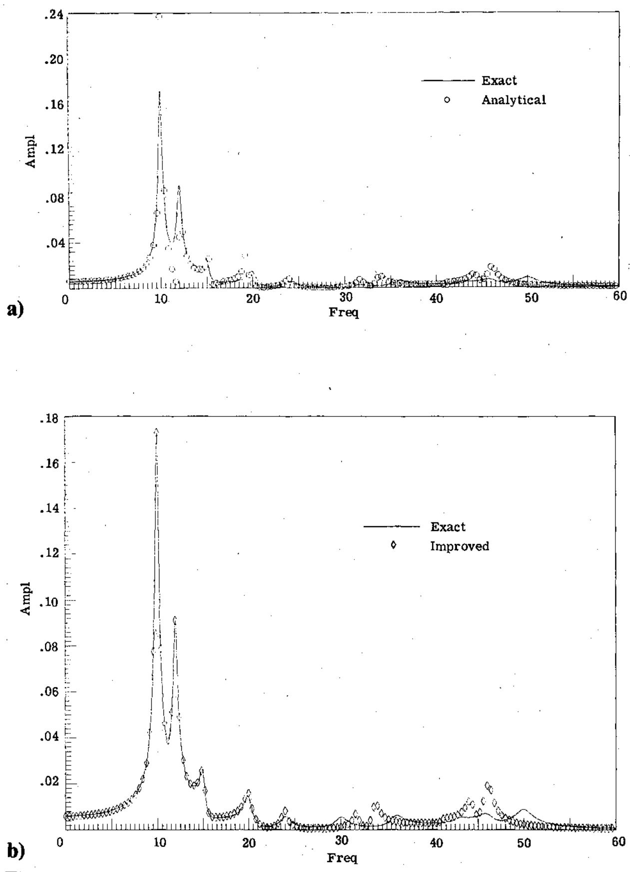

Fig. 1 Impulse responses of station four: a) analytical model b) improved model.

Results and Discussion

Table 1 shows the exact, analytical, and improved natural frequencies and damping factors. The second and fourth mode shapes for the exact, analytical, and improved mathematical models are listed in Table 2 for comparison. Figures 1a and 1b show the fourth station response of the analytical model and improved model, respectively, plotted on the exact response due to an impulse at the first station. These figures show responses over the whole frequency range (0-60.0 Hz). While noticeable improvements are produced over the correction frequency range (0-20.0 Hz), no adverse effects resulted from the improvement process over the remainder of the frequency range.

Conclusions

A direct technique to use experimental and analytical modal parameters to improve an existing analytical model is presented. The corrected model's response resembles the exact model's response more accurately than the analytical model over the frequency range of the measured modal test data. No adverse effects on the improved system's responses resulted beyond the correction frequency range. Built into the algorithm procedure is the computation of the normal modes, corresponding to the set of measured complex modes, that are used for mass matrix correction. This feature promises to eliminate the errors that may result from using approximated complex modes as normal modes for mass matrix correction, which makes this technique advantageous when dealing with complex structures possessing, not necessarily high levels of damping, but a high degree of nonproportionality in damping.

Acknowledgment

This work was supported by a grant from NASA Langley Research Center.

References

[1]Young, J. P. and On, F. J., "Mathematical Modeling Via Direct Use of Vibration Data," SAE National Aeronautic and Space Engineering and Manufacturing Meeting, Los Angeles, Calif., Oct. 1969.

[2]Thoren, A. R., "Derivation of Mass and Stiffness Matrices from Dynamic Test Data," *Proceedings of the AIAA/ASME/SAE 13th SDM Conference*, San Antonio, Tex., April 1972.

[3]Berman, A., "System Identification of a Complex Structure," *Proceedings of the AIAA/ASME/SAE 16th SDM Conference*, Denver, Colo., May 1975.

[4]Potter, R. and Richardson, M., "Mass Stiffness and Damping Matrices from Measured Modal Parameters," International Instrumentation-Automation Conference, New York, Oct. 1974.

[5]Berman, A., "Mass Matric Correction Using and Incomplete Set of Measured Modes," *AIAA Journal*, Vol. 17, Oct. 1979.

[6]Chen, J. C. and Garba, J. A., "Matrix Perturbation for Analytical Model Improvement," *Proceedings of the AIAA/ASME/ASCE/AHS 20th SDM Conference*, St. Louis, Mo., April 1979.

[7]Baruch, M., "Optimization Procedure to Correct Stiffness and Flexibility Matrices Using Vibration Tests," *AIAA Journal*, Vol. 16, April 1978.

[8]Berman, A., Wei, F. S., and Rao, K. W., "Improvement of Analytical Models Using Modal Test Data," *Proceedings of the AIAA/ASME/ASCE/AHS 21st SDM Conference*, Seattle, Wash., May 1980.

[9]Meirovitch, L., *Analytical Methods in Vibrations*, McMillan Publishing Co., 1967.

[10]Lewis, R. C. and Wrisley, D. L., "A System for the Excitation of Pure Natural Modes of Complex Structures," *Journal of the Aerospace Sciences*, Vol. 17, 1950, pp. 705-722.

[11]Leppert, E. L., Lee, S. H., Day, F. D., Chapman, P. C., and Wada, B. K., "Comparison of Modal Test Results: Multipoints Sine Versus Single Point Random," ASE Paper 760879, Nov. 1976.

[12]Chu, F. H., Voorhees, C., Metzger, W. W., and Wilding, R., "Spacecraft Modal Testing Using Systematic Multi-Shaker Sine-Dwell Testing Techniques," *Shock and Vibration Bulletin 51*, Pt. 2, May 1981, pp. 41-58.

[13]Ibrahim, S. R., "Computation of Normal Modes from Identified Complex Modes," *AIAA Journal*, Vol. 21, No. 3, March 1983, pp. 446-451.